4 Classification of Second-Order Equations

4.1 Types of Second-Order Equations

We now turn our attention to second-order equations

$$F(\vec{x}, u, Du, D^2u) = 0.$$

In general, higher-order equations are more complicated to solve than first-order equations. Consequently, we will only be studying linear equations. First, let's consider a second-order equation of only two independent variables. We will then discuss second-order equations in higher dimensions. Consider a linear, second-order equation of the form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \tag{4.1}$$

In studying second-order equations, it has been shown that solutions of equations of the form (4.1) have different properties depending on the coefficients of the highest-order terms, a,b,c. We will classify these equations into three different categories. If $b^2 - 4ac > 0$, we say the equation is **hyperbolic**. If $b^2 - 4ac = 0$, we say the equation is **parabolic**. If $b^2 - 4ac < 0$, we say the equation is **elliptic**.

Example 1.

- The wave equation
$$u_{tt} - u_{xx} = 0 \qquad \text{is hyperbolic.}$$
- The Laplace equation
$$u_{xx} + u_{yy} = 0 \qquad \text{is elliptic.}$$
- The heat equation
$$u_t - u_{xx} = 0 \qquad \text{is parabolic.}$$

◇

4.2 Canonical Form.

The three equations in Example 1 above are of particular interest not only because they are derived from physical principles, but also because every second-order linear equation of the form (4.1) can be reduced to an equation of one of those forms (plus lower-order terms) by making a change of variables.

Theorem 2. (Ref: Strauss, Sec. 1.6) *By a linear transformation of the independent variables, any equation of the form*

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

can be reduced to one of the following forms.

1. Elliptic case: *If $b^2 - 4ac < 0$, the equation is reducible to*

$$u_{xx} + u_{yy} + \ldots = 0.$$

2. Hyperbolic case: *If $b^2 - 4ac > 0$, the equation is reducible to*

$$u_{xx} - u_{yy} + \ldots = 0.$$

3. Parabolic case: *If $b^2 - 4ac = 0$, the equation is reducible to*

$$u_{xx} + \ldots = 0,$$

where $\ldots$ represent lower-order terms.

Proof. Consider the hyperbolic case. Without loss of generality, we may assume $a = 1$, $d = e = f = 0$. By completing the square, we can write the equation as

$$\left(\partial_x + \frac{b}{2}\partial_y\right)^2 u - \left(\left(\frac{b}{2}\right)^2 - c\right)\partial_y^2 u = 0.$$

Now if we introduce new variables ξ, η such that

$$\begin{aligned} \partial_\xi &= \partial_x + \frac{b}{2}\partial_y \\ \partial_\eta &= \sqrt{(b/2)^2 - c}\,\partial_y \end{aligned} \tag{4.2}$$

then our equation will become

$$u_{\xi\xi} - u_{\eta\eta} = 0,$$

which is a hyperbolic equation of the form described. Now we first note that by the assumption that $b^2 - 4ac > 0$ and $a = 1$, $(b/2)^2 - c > 0$ so $\sqrt{(b/2)^2 - c}$ is defined.

To make this change of variables, we introduce variables ξ, η such that

$$\begin{aligned} x_\xi &= 1 \qquad & x_\eta &= 0 \\ y_\xi &= \frac{b}{2} \qquad & y_\eta &= \sqrt{(b/2)^2 - c}. \end{aligned}$$

That is,

$$\begin{aligned} x &= \xi \\ y &= (b/2)\xi + (\sqrt{(b/2)^2 - c})\eta \end{aligned}$$

which implies

$$\begin{aligned} \xi &= x \\ \eta &= [(b/2)^2 - c]^{-1/2}[-(b/2)x + y]. \end{aligned}$$

With this change of variables, the operators ∂_ξ, and ∂_η will satisfy (4.2), and our equation will take the form

$$u_{\xi\xi} - u_{\eta\eta} = 0,$$

as desired.

If $a = 0$ above, our equation has the form

$$bu_{xy} + cu_{yy} = 0.$$

The assumption that $b^2 - 4ac > 0$ implies $b \neq 0$ (if $a = 0$). This equation can be written as

$$\partial_y[b\partial_x + c\partial_y]u = 0.$$

Introduce variables ξ and η such that

$$\begin{aligned} \partial_\xi &= b\partial_x + c\partial_y \\ \partial_\eta &= \partial_y. \end{aligned}$$

That is, letting

$$\begin{aligned} x_\xi &= b \qquad & x_\eta &= 0 \\ y_\xi &= c \qquad & y_\eta &= 1 \end{aligned}$$

which implies

$$\begin{aligned} x &= b\xi \\ y &= c\xi + \eta \end{aligned}$$

or

$$\begin{aligned} \xi &= (1/b)x \\ \eta &= -(c/b)x + y, \end{aligned}$$

our equation becomes

$$u_{\xi\eta} = 0. \tag{4.3}$$

We note that an equation of the form (4.3) is sometimes also known as the canonical form for hyperbolic equations in two spatial dimensions. In particular, any equation of the form (4.3) can easily be transformed to the standard form by introducing variables $\widetilde{\xi}, \widetilde{\eta}$ such that

$$\begin{aligned} \partial_\xi &= \partial_{\widetilde{\xi}} + \partial_{\widetilde{\eta}} \\ \partial_\eta &= \partial_{\widetilde{\xi}} - \partial_{\widetilde{\eta}}. \end{aligned}$$

That is,

$$\begin{aligned} \widetilde{\xi}_\xi &= 1 \qquad & \widetilde{\xi}_\eta &= 1 \\ \widetilde{\eta}_\xi &= 1 \qquad & \widetilde{\eta}_\eta &= -1 \end{aligned}$$

which implies

$$\begin{aligned}\widetilde{\xi} &= \xi + \eta \\ \widetilde{\eta} &= \xi - \eta.\end{aligned}$$

As a result our equation takes on the form

$$(\partial_{\widetilde{\xi}} + \partial_{\widetilde{\eta}})(\partial_{\widetilde{\xi}} - \partial_{\widetilde{\eta}})u = 0,$$

or

$$u_{\widetilde{\xi}\widetilde{\xi}} - u_{\widetilde{\eta}\widetilde{\eta}} = 0,$$

as desired.

The elliptic and parabolic cases can be proven similarly. □

4.3 Generalizing to Higher Dimensions

We now generalize the definitions of ellipticity, hyperbolicity, and parabolicity to second-order equations in n dimensions. Consider the second-order equation

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = 0. \tag{4.4}$$

Assuming that the mixed partial derivatives are equal, we may as well assume that $a_{ij} = a_{ji}$. For example, the equation

$$u_{xx} + 2u_{xy} - 3u_{yz} + 5u_{zz} = 0$$

can be written as

$$u_{xx} + u_{xy} + u_{yx} - (3/2)u_{yz} - (3/2)u_{zy} + 5u_{zz} = 0.$$

Claim 3. *By making an appropriate change of variables, we can write the top-order term*

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} \tag{4.5}$$

as

$$\sum_{k=1}^{n} d_k u_{x_k x_k}$$

where the coefficients d_k are the eigenvalues of the $n \times n$ matrix $A = (a_{ij})$. Moreover, by making a change of scale, we can choose the coefficients $d_i = 0, \pm 1$ for $i = 1, \ldots, n$.

Proof. Let $\vec{x} = [x_1 \ldots x_n]^T$. Below, we will find an appropriate $n \times n$ matrix B such that by defining

$$\vec{\xi} = B\vec{x},$$

we can write (4.5) in the form

$$d_1 u_{\xi_1\xi_1} + d_2 u_{\xi_2\xi_2} + \ldots + d_n u_{\xi_n\xi_n}$$

where $d_i = 0, \pm 1$.

Letting

$$\vec{\xi} = B\vec{x},$$

we have

$$\xi_k = \sum_{m=1}^{n} b_{km} x_m,$$

and, therefore,

$$\begin{aligned} \frac{\partial}{\partial x_i} &= \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \xi_k} \\ &= \sum_{k=1}^{n} b_{ki} \frac{\partial}{\partial \xi_k}. \end{aligned}$$

Consequently,

$$\begin{aligned} u_{x_i x_j} &= \left(\sum_{l=1}^{n} b_{lj} \frac{\partial}{\partial \xi_l} \right) \left(\sum_{k=1}^{n} b_{ki} \frac{\partial}{\partial \xi_k} \right) u \\ &= \sum_{l,k} \frac{\partial^2}{\partial \xi_l \partial \xi_k} b_{lj} b_{ki} u \\ &= \sum_{l,k} b_{lj} b_{ki} u_{\xi_l \xi_k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i,j} a_{ij} u_{x_i x_j} &= \sum_{i,j} \left(\sum_{l,k} b_{lj} a_{ij} b_{ki} u_{\xi_l \xi_k} \right) \\ &= \sum_{l,k} \left(\sum_{i,j} b_{lj} a_{ij} b_{ki} \right) u_{\xi_l \xi_k}. \end{aligned}$$

Now, letting

$$d_{kl} \equiv \sum_{i,j} b_{lj} a_{ij} b_{ki},$$

we have

$$\sum_{i,j} a_{ij} u_{x_i x_j} = \sum_{l,k} d_{kl} u_{\xi_l \xi_k}.$$

We claim that for an appropriate choice of B, we can make

$$d_{kl} = \begin{cases} \pm 1 & k = l \\ 0 & k \neq l. \end{cases}$$

Let $A = (a_{ij})$, the coefficient matrix associated with (4.5). By the assumption that $a_{ij} = a_{ji}$, we know that A is a symmetric matrix. Let $(C)_{lk}$ denote the entry in row l, column k of the matrix C. Therefore, we have

$$\begin{aligned}
d_{kl} &= \sum_{i,j=1}^{n} b_{lj} a_{ij} b_{ki} \\
&= \sum_{j=1}^{n} b_{lj} \left(\sum_{i=1}^{n} a_{ij} b_{ki} \right) \\
&= \sum_{j=1}^{n} b_{lj} \left(\sum_{i=1}^{n} a_{ji} b_{ki} \right) \\
&= \sum_{j=1}^{n} b_{lj} (AB^T)_{jk} \\
&= (BAB^T)_{lk} \\
&= ((BAB^T)^T)_{kl} \\
&= (BA^T B^T)_{kl} \\
&= (BAB^T)_{kl}.
\end{aligned}$$

So, if we can choose B such that BAB^T is diagonal, then we would have $d_{kl} = 0$ for $l \neq k$. Again, using the assumption that A is symmetric, we know there exists an orthogonal matrix S such that $S^T AS = D$ where D is a diagonal matrix whose diagonal entries are just the eigenvalues of A. (S is the matrix whose columns form an orthonormal eigenbasis for A.) Letting $B = S^T$, we have $d_{kl} = 0$ for $k \neq l$. Therefore, by making the change of variables $\vec{\xi} = B\vec{x}$, we can write our top-order term (4.5) as

$$\sum_{k=1}^{n} d_k u_{\xi_k \xi_k},$$

where the d_k are the eigenvalues of A. In order to make the coefficients $d_k = \pm 1$, we just need to make a change of scale in our variables ξ_k, i.e. - let $\widetilde{\xi_k} = (1/\sqrt{|d_k|})\xi_k$.

$\square$

In the above claim, we have shown that every linear, second-order equation of the form (4.4) can be written in the canonical form

$$\sum_{k=1}^{n} d_k u_{x_k x_k} + \ldots = 0,$$

where the coefficients d_k are the eigenvalues of A and $''\ldots''$ represents lower-order terms. Moreover, by a change of variables, we may choose the $d_k = 0, \pm 1$.

Extending the definitions we gave previously for second-order equations in two spatial variables, we say an equation of the form (4.4) is **elliptic** if the eigenvalues of $A = (a_{ij})$ are all positive or all negative. In particular, by the above claim, elliptic equations can all be written in the canonical form

$$\sum_{i=1}^{n} u_{x_i x_i} + \ldots = 0.$$

We say an equation of the form (4.4) is **hyperbolic** if none of the eigenvalues are zero and one of them has the opposite sign of the $(n-1)$ others. By the above claim, all hyperbolic equations can be written in the canonical form,

$$u_{x_1 x_1} - \sum_{i=2}^{n} u_{x_i x_i} + \ldots = 0.$$

We say an equation of the form (4.4) is **parabolic** if exactly one of the eigenvalues is zero and all the others have the same sign. By the above claim, all parabolic equations can be written in the canonical form,

$$\sum_{i=2}^{n} u_{x_i x_i} + \ldots = 0.$$